

Toroidal Perturbations of Friedmann-Robertson-Walker Universes

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Explicit expressions are found for the axisymmetric metric perturbations of the closed, flat and open FRW universes caused by toroidal motions of the cosmic fluid. The perturbations are decomposed in vector spherical harmonics on 2-spheres, but the radial dependence is left general. Solutions for general odd-parity l -pole perturbations are given for either angular velocities or angular momenta prescribed. In particular, in case of closed universes the solutions require a special treatment of the Legendre equation.

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I. INTRODUCTION

There exists a class of perturbations in the Newtonian astrophysics which represent differential rotations of a spherical star in hydrostatic equilibrium. The displacement fields are time-independent and have no radial components [1]:

$$\begin{aligned} 4\pi G\rho\xi_r &= 0, & 4\pi G\rho\xi_\theta &= \frac{T_{lm}(r)}{r\sin\theta} \frac{\partial Y_{lm}}{\partial\varphi}, \\ 4\pi G\rho\xi_\varphi &= -\frac{T_{lm}(r)}{r} \frac{\partial Y_{lm}}{\partial\theta}, \end{aligned} \quad (1.1)$$

where T_{lm} is any function satisfying the condition that ξ_θ and ξ_φ are finite. These solutions define the *toroidal perturbations*. When $l = 1$, the displacements can be written as

$$\xi = \delta\Omega(r) \times \mathbf{r}, \quad |\delta\Omega| = \frac{1}{4\pi G\rho^2} T_{ml}, \quad (1.2)$$

which represent rigid rotations of the internal spherical surfaces around the center. If $\delta\Omega$ does not depend on r , the whole star is rotated as a rigid body. To get a uniform slow rotation, rather than a static displacement, we assume equation (1.1) to be true with ξ replaced by the Eulerian velocity field; then displacements are linear in time. Such perturbations are analogues of the cosmological toroidal perturbations investigated in this paper.

The toroidal perturbations (1.1) are transverse and divergence free. They are often called the “trivial modes”. The trivial modes become important when one considers the second order perturbations ($\simeq [\delta\Omega]^2$) [2]. They develop into the so-called r -modes in non-radial oscillations of a rotating star (see e.g. [3]). They also give rise to non-zero eigenvalues when an external (e.g. magnetic) field is added.

Toroidal oscillations are important in geophysics [4]. Non-rotating, spherically symmetric Earth models admit “trivial toroidal modes” which are associated with vanishing eigenfrequencies; they do not alter the elastic-gravitational potential of the Earth. As in astrophysics,

their counterparts play a significant role on a rotating Earth. In the solid core the toroidal modes are nontrivial even in spherical models: the presence of an anisotropic stress tensor produces a traction.

The velocity pattern of the axisymmetric $l = 2$ toroidal (quadrupole) mode is shown in Fig. 1. The individual rings of fluid rotate independently, preserving their local angular momentum. For non-trivial modes in the solid core the displacements will reverse after a half of the oscillation cycle.

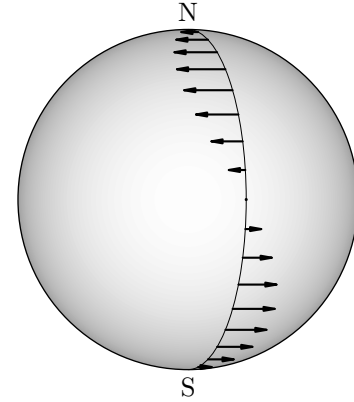


FIG. 1: Velocity field for axisymmetric quadrupole ($l = 2$) toroidal perturbations.

In general relativity even “trivial modes” in spherical systems become “non-trivial” since they produce dragging of inertial frames and, hence, influence essentially gravitational field.

In the following we investigate the toroidal perturbations of Friedmann-Robertson-Walker (FRW) universes of all three types ($k = -1, 0, +1$). We confine ourselves to the axisymmetric case. Since the backgrounds admit homogeneous and isotropic foliations, non-symmetric perturbations can be found from the axisymmetric ones.

Indeed, in any spherically symmetric background system, one can restrict oneself to axisymmetric modes of perturbations since non-axisymmetric modes with an

$e^{im\varphi}$ dependence ($m = -l, \dots, +l$) can be derived from the axisymmetric perturbations ($m = 0$) by suitable rotation of the axes. If, in the original coordinates, the new polar axis will be pointing in a direction (θ', φ') , then an axisymmetric mode with respect to the new axes can be decomposed in non-axisymmetric modes in the original axes by using the addition theorem for spherical harmonics (see [7], p. 138–139 for more details). Each the $e^{im\varphi}$ component will separately solve the perturbation equation with the original radial functions. Since we are studying vector perturbations, the vector components will also transform under a rotation. (So, in (1.1) with an axisymmetric mode only $\xi_\varphi \neq 0$, but a general rotation will induce $\xi_\theta \neq 0$ as well.)

The equations to be solved are derived and discussed in Section 2 of [5], hereafter Paper I. Here we just summarize few basic relations. We start out from the perturbed FRW metric in the form

$$\begin{aligned} ds^2 &= (\bar{g}_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= dt^2 - a^2(t)f_{ij}dx^i dx^j + h_{\mu\nu}dx^\mu dx^\nu, \end{aligned} \quad (1.3)$$

where the background metric $\bar{g}_{\mu\nu}$ is used to move indices; the time-independent part of the spatial background metric f_{ij} [$i, j, k = 1, 2, 3$] is used to define the 3-covariant derivative ∇_k , $\nabla^k = f^{kl}\nabla_l$. We choose spatial harmonic gauge conditions on $t = \text{constant}$ slices. The slices themselves are chosen so that the perturbation of the external curvature vanishes (uniform Hubble expansion gauge) – see Paper I for details. In addition, since we are interested in the toroidal perturbations we also assume the perturbation of the fluid velocity to be transverse, $\nabla_k V^k = 0$, and the same for the metric vector perturbation h_{0k} , $\nabla^k h_{0k} = 0$.

In the axisymmetric case we get the perturbed constraint equation

$$\nabla^2 h_{0\varphi} + 2kh_{0\varphi} = 2a^2\kappa\delta T_\varphi^0, \quad (1.4)$$

in which $\nabla^2 = f^{kl}\nabla_k\nabla_l$ and the perturbed Bianchi identities (the dot denoting $\partial/\partial t$)

$$\left[a^3(\rho + p)(a^2 r^2 \sin^2 \theta V^\varphi - h_{0\varphi}) \right]^\cdot = 0, \quad (1.5)$$

equivalently

$$\left[a^3 \delta T_\varphi^0 \right]^\cdot = 0. \quad (1.6)$$

These just express the conservation of angular momentum of each element of the axially symmetric ring of fluid, analogously to the “trivial modes” in astrophysics mentioned above. To have solutions in more closed forms, we decompose quantities only in vector spherical harmonics in variables θ, φ , but leave the radial dependence general. Hence, we write

$$h_{0\varphi} = a^2 r^2 \sum_{l=1}^{\infty} \omega_l(t, r) \sin \theta Y_{l0,\theta}, \quad (1.7)$$

$$V_\varphi = -a^2 r^2 \sum_{l=1}^{\infty} \Omega_l(t, r) \sin \theta Y_{l0,\theta}, \quad (1.8)$$

and

$$\begin{aligned} \delta T_\varphi^0 &= a^2(\rho + p)r^2 \sum_{l=1}^{\infty} (\omega_l - \Omega_l) \sin \theta Y_{l0,\theta} \\ &= \sum_{l=1}^{\infty} [\delta T_\varphi^0(t, r)]_l \sin \theta Y_{l0,\theta}. \end{aligned} \quad (1.9)$$

From equation (1.4) then follows the radial equation for each l in the form

$$\begin{aligned} & -\sqrt{1 - kr^2} \frac{1}{r^2} \frac{\partial}{\partial r} \left[\sqrt{1 - kr^2} \frac{\partial}{\partial r} (r^2 \omega_l) \right] + \\ & \frac{l(l+1)}{r^2} \omega_l - 4k\omega_l = \lambda^2 (\Omega_l - \omega_l), \end{aligned} \quad (1.10)$$

where $\lambda^2 = 2\kappa a^2(\rho + p) = 4(k - a^2 \dot{H})$, $H = \dot{a}/a$ is the Hubble constant; the last relation follows from the background equations for arbitrary ρ, p, k and cosmological constant Λ .

For $k = 0$ equation (1.10) can be written with the angular momentum density $(\delta T_\varphi^0)_l$ as a source in the form

$$\begin{aligned} \frac{1}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial \omega_l}{\partial r} \right) - \frac{l(l+1) - 2}{r^2} \omega_l &= \lambda^2 (\omega_l - \Omega_l) \\ &= \frac{2\kappa}{r^2} (\delta T_\varphi^0)_l, \end{aligned} \quad (1.11)$$

for the fluid angular velocity as a source,

$$\begin{aligned} \frac{1}{r^4} \frac{\partial}{\partial r} \left(r^4 \frac{\partial \omega_l}{\partial r} \right) - \left[\lambda^2 + \frac{l(l+1) - 2}{r^2} \right] \omega_l \\ = -\lambda^2 \Omega_l; \end{aligned} \quad (1.12)$$

here $\lambda^2 = -4a^2 \dot{H} = 2\kappa a^2(\rho + p)$.

For $k = \pm 1$ it is advantageous to write $r^2 = k(1 - \mu^2)$, or $\mu = \sqrt{1 - kr^2}$, to obtain

$$\begin{aligned} \frac{1}{[k(1 - \mu^2)]^{3/2}} \frac{\partial}{\partial \mu} \left\{ [k(1 - \mu^2)]^{5/2} \frac{\partial \omega_l}{\partial \mu} \right\} - \frac{l(l+1) - 2}{k(1 - \mu^2)} \omega_l \\ = \frac{2\kappa}{k(1 - \mu^2)} (\delta T_\varphi^0)_l. \end{aligned} \quad (1.13)$$

If we set

$$\omega_l = [k(1 - \mu^2)]^{-3/4} \bar{\omega}_l, \quad (1.14)$$

equation (1.13) becomes the Legendre equation for $\bar{\omega}_l$ with $(\delta T_\varphi^0)_l$ as the source:

$$\begin{aligned} \frac{\partial}{\partial \mu} \left[k(1 - \mu^2) \frac{\partial \bar{\omega}_l}{\partial \mu} \right] + \left[k \frac{3}{2} \left(\frac{3}{2} + 1 \right) - \frac{(l + \frac{1}{2})^2}{k(1 - \mu^2)} \right] \bar{\omega}_l \\ = \frac{2\kappa}{[k(1 - \mu^2)]^{1/4}} (\delta T_\varphi^0)_l. \end{aligned} \quad (1.15)$$

With the fluid angular velocity Ω_l as the source, the equation reads

$$\begin{aligned} \frac{\partial}{\partial \mu} \left[k(1 - \mu^2) \frac{\partial \bar{\omega}_l}{\partial \mu} \right] + \left[k\nu(\nu + 1) - \frac{(l + \frac{1}{2})^2}{k(1 - \mu^2)} \right] \bar{\omega}_l \\ = -K_l \equiv -\lambda^2 \Omega_l [k(1 - \mu^2)]^{3/4}, \end{aligned} \quad (1.16)$$

where

$$\left(\nu + \frac{1}{2}\right)^2 = 4 - 2k\kappa a^2(\rho + p) = 4 - k\lambda^2 = 4ka^2\dot{H}. \quad (1.17)$$

In the following we shall first solve equations (1.11) and (1.12) for flat universes (Section 2). The interesting but more complicated case is that of closed universes, i.e., solutions of equations (1.15) and (1.16) with $k = +1$. We shall discover that Legendre functions $Q_{\nu=3/2}^{l+\frac{1}{2}}$ which solve equation (1.15) and are regular at least at one pole on the 3-sphere, in fact vanish identically for $l \geq 2$! However, appropriate solutions can be found by considering the derivatives of the Legendre functions with respect to their degree ν . We did not find this fact in the literature on special functions. We analyze solutions of both equation (1.15) with the angular momentum given and (1.16) with the fluid angular velocity given for closed universes in Section 3. Finally, in Section 4, we study them in detail also for open universes. Roughly speaking, the dragging effects are not damped when angular momenta are given as their sources, whereas they are, if fluid's angular velocity is considered as a source. The physical explanation of this apparent paradox is the same as that given in Paper I for $l = 1$ perturbations.

Before we turn to solving the equations, we wish to emphasize that all our solutions are true at any given instant of time. However, the angular momentum and angular velocity distributions are related to those at earlier times by equations of motion (i.e., by the contracted Bianchi identities). In our case of toroidal perturbations they lead back to local conservation of angular momentum density, which demonstrates the advantage of taking angular momentum as the source. The time evolution of the dragging is thus independent of l . It is analyzed in Section 5 of Paper I.

II. AXISYMMETRIC TOROIDAL PERTURBATIONS FOR $k = 0$

A. The angular momentum as a source of ω

The homogeneous equation corresponding to (1.11) with $(\delta T_\varphi^0)_l$ given is

$$\omega_l'' + \frac{4}{r}\omega_l' - \frac{l(l+1)-2}{r^2}\omega_l = 0. \quad (2.1)$$

This can be easily solved by writing $\omega_l = r^\alpha \bar{\omega}_l$, and choosing α so that the coefficient of $\bar{\omega}_l$ vanishes. It implies $\alpha = l-1$ or $\alpha = -(l+2)$ and the first-order equation for $\bar{\omega}_l'$. Two simple integrations lead to the solutions

$$\omega_l^{(I)} = A_l r^{l-1}, \quad \omega_l^{(II)} = B_l r^{-(l+2)}, \quad (2.2)$$

A_l, B_l constants. The Wronskian is $\mathcal{W}_l = A_l B_l (2l+1)r^{-4}$. The solution of the inhomogeneous equation

(1.11) which decays at infinity and is well-behaved at the origin is given by

$$\omega_l = \frac{-2\kappa}{2l+1} \left[\frac{1}{r^{l+2}} \int_0^r (r')^{l+1} (\delta T_\varphi^0)_l dr' + r^{l-1} \int_r^\infty (r')^{-l} (\delta T_\varphi^0)_l dr' \right]. \quad (2.3)$$

In particular for the dipole ($l = 1$) perturbations

$$\omega_{l=1} = -\frac{16\pi}{3} \left[\frac{1}{r^3} \int_0^r (r')^2 (\delta T_\varphi^0)_1 dr' + \int_r^\infty \frac{1}{r'} (\delta T_\varphi^0)_1 dr' \right]. \quad (2.4)$$

If we write

$$\delta T_\varphi^0 = (\delta T_\varphi^0)_1 \sin \theta (-\sqrt{3/4\pi} \sin \theta), \quad (2.5)$$

and integrate over θ , we find that (2.4) coincides with the second equation in (I.3.4); hereafter, we denote in this way equations in Paper I.

Equation (2.3) demonstrates how for general l -pole distributions, “local” angular momenta, determined by $(\delta T_\varphi^0)_l$, at all distances contribute instantaneously to the dragging of inertial frames without any exponential cut-off or influence of any horizon – see the detailed discussion of this point for $l = 1$ perturbations in Paper I. Notice that angular momenta $(\delta T_\varphi^0)_l$ for $l > 1$ do not, however, contribute to the *total* angular momentum in a spherical layer $\langle r_1, r_2 \rangle$ given by [cf. (I.2.24) and (I.2.25)]

$$J(r_1, r_2) = -2\pi \int_{r_1}^{r_2} dr \int_0^\pi d\theta a^3 r^2 \sin \theta \delta T_\varphi^0. \quad (2.6)$$

Substituting for δT_φ^0 from equation (1.9), integrating by parts over θ and realizing that

$$\int_0^\pi Y_{l0} \cos \theta \sin \theta d\theta = \sqrt{4\pi/3} \delta_{l1} \quad (2.7)$$

due to the orthogonality of spherical harmonics, expression (2.6) reduces to

$$\begin{aligned} J(r_1, r_2) &= 4\sqrt{\pi/3} a^3 \int_{r_1}^{r_2} dr r^2 (\delta T_\varphi^0)_{l=1} \\ &= 4\sqrt{\pi/3} a^5 (\rho + p) \int_{r_1}^{r_2} dr r^4 (\omega_{l=1} - \Omega_{l=1}). \end{aligned} \quad (2.8)$$

Such result is understandable on intuitive grounds: e.g. for $l = 2$ odd-parity perturbations rotations at antipodal θ 's are equal and opposite – the “northern” hemisphere rotates in the opposite direction to the “southern” hemisphere (see Fig. 1).

B. The fluid angular velocity as a source

The equation to be solved is (1.12). We introduce $z = \lambda r$ and assume $\lambda > 0$, so $z \geq 0$. Putting then

$$\omega_l = z^{-3/2} \bar{\omega}_l, \quad (2.9)$$

the homogeneous part of (1.12) turns into the equation

$$\frac{d^2 \bar{\omega}_l}{dz^2} + \frac{1}{z} \frac{d \bar{\omega}_l}{dz} - \left[1 + \frac{(l + \frac{1}{2})^2}{z^2} \right] \bar{\omega}_l = 0. \quad (2.10)$$

This is the modified Bessel equation, the general solution being given by

$$\bar{\omega}_l = C_l I_{l+\frac{1}{2}}(z) + D_l K_{l+\frac{1}{2}}(z), \quad (2.11)$$

where $I_{l+\frac{1}{2}}$ and $K_{l+\frac{1}{2}}$ are modified Bessel functions. For an integer l these functions are given explicitly by finite sums (see e.g. 8.467, 8.468 in [9], for still shorter expressions in terms of $(d/dz)^l$, see [10], [11]) which imply the following asymptotic expansions at $z \rightarrow \infty$,

$$\begin{aligned} I_{l+\frac{1}{2}}(z) &= \frac{e^z}{\sqrt{2\pi z}} + \dots, \\ K_{l+\frac{1}{2}}(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} + \dots, \end{aligned} \quad (2.12)$$

so, independent of l in the leading terms at infinity. At the origin $z \rightarrow 0$,

$$\begin{aligned} I_{l+\frac{1}{2}}(z) &= \sqrt{\frac{2}{\pi}} \frac{1}{(2l+1)!!} z^{l+\frac{1}{2}} \left[1 + \frac{1}{2(2l+3)} z^2 + \dots \right], \\ K_{l+\frac{1}{2}}(z) &= \sqrt{\frac{\pi}{2}} \frac{(2l)!}{2^l l!} \frac{1}{z^{l+\frac{1}{2}}} + \dots. \end{aligned} \quad (2.13)$$

The general solution of the original homogeneous equation is thus given by $\omega_l = \omega_l^{(I)} + \omega_l^{(II)}$, where

$$\omega_l^{(I)} = A_l \bar{I}_{l+\frac{1}{2}}(z) = A_l z^{-3/2} I_{l+\frac{1}{2}}(z), \quad (2.14)$$

$$\omega_l^{(II)} = B_l \bar{K}_{l+\frac{1}{2}}(z) = B_l z^{-3/2} K_{l+\frac{1}{2}}(z), \quad (2.15)$$

($\bar{I}_{l+\frac{1}{2}}$ and $\bar{K}_{l+\frac{1}{2}}$ are defined by these relations). $\omega_l^{(I)}$ behaves well at $z \rightarrow 0$, $\omega_l^{(II)}$ at $z \rightarrow \infty$. It is easy to see that for $l = 1$ one obtains the asymptotic expansions given in Paper I. The Wronskian reads (for the Wronskian of $I_{l+\frac{1}{2}}$ and $K_{l+\frac{1}{2}}$ see [10], p. 68)

$$W_l(z) = -A_l B_l z^{-4}. \quad (2.16)$$

Using again the method of variation of parameters to find the solution of the inhomogeneous equation (1.12) which is well behaved at $z \rightarrow 0$ and vanishes at $z \rightarrow \infty$, we arrive at

$$\begin{aligned} \omega_l &= \bar{K}_{l+\frac{1}{2}}(z) \int_0^z (z')^4 \bar{I}_{l+\frac{1}{2}}(z') \Omega_l(z') dz' \\ &+ \bar{I}_{l+\frac{1}{2}}(z) \int_z^\infty (z')^4 \bar{K}_{l+\frac{1}{2}}(z') \Omega_l(z') dz', \end{aligned} \quad (2.17)$$

$\bar{I}_{l+\frac{1}{2}}$ and $\bar{K}_{l+\frac{1}{2}}$ are defined by (2.14) and (2.15).

The solution (2.17) for general l can be explored in a way similar to that for $l = 1$, as was done in Paper I. So, to find ω for small z with sources Ω_l not close, we neglect the first term in (2.17) and expand $\bar{I}_{l+\frac{1}{2}}$ in the second term:

$$\begin{aligned} \omega_l(z) &= \sqrt{\frac{2}{\pi}} \frac{1}{(2l+1)!!} z^{l-1} \left[1 + \frac{1}{2(2l+3)} z^2 + \dots \right] \times \\ &\times \int_z^\infty (z')^4 \bar{K}_{l+\frac{1}{2}}(z') \Omega_l(z') dz'. \end{aligned} \quad (2.18)$$

For the source Ω_l at $z' \gg 1$ we expand $\bar{K}_{l+\frac{1}{2}}$ [see (2.12)]:

$$\begin{aligned} \omega_l(z) &= \frac{1}{(2l+1)!!} z^{l-1} \left[1 + \frac{1}{2(2l+3)} z^2 + \dots \right] \times \\ &\times \int_z^\infty (z')^2 e^{-z'} \Omega_l(z') dz'. \end{aligned} \quad (2.19)$$

If, in addition, the source is localized in a small interval $r_0(1 \pm \Delta)$, $\Delta \ll 1/\lambda$, we find

$$\begin{aligned} \omega_l(r) &= \frac{1}{(2l+1)!!} (\lambda r)^{l-1} \left[1 + \frac{1}{2(2l+3)} (\lambda r)^2 \right] \times \\ &\times (\lambda r_0)^3 e^{-\lambda r_0} \bar{\Omega} 2\Delta. \end{aligned} \quad (2.20)$$

This generalizes equation (I.4.7) for any l . Thus, we recover the exponential decay which is now seen to be independent of l , but we also discover the power-law decline, given by $(\lambda r)^{l-1}/(2l+1)!!$ ($\lambda r \ll 1$) which increases with increasing l . When Ω_l is concentrated near z_0 , in $z_0 \pm \lambda\Delta$, then close to $z_0 \gg 1$ and with $\Omega_l = \bar{\Omega}_l$ we find $\omega_l(z_0) = \lambda\Delta \bar{\Omega}_l$, as in Paper I.

III. AXISYMMETRIC TOROIDAL PERTURBATIONS FOR $k = 1$

We wish to solve the inhomogeneous equation (1.13) for ω_l which, after the substitution (1.14), turns into the inhomogeneous equation corresponding either to (1.15) or (1.16).

So, to find solutions we have first to explore appropriate solutions of the homogeneous Legendre equation

$$\frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \bar{\omega}_l}{\partial \mu} \right] + \left[\nu(\nu+1) - \frac{(l + \frac{1}{2})^2}{1 - \mu^2} \right] \bar{\omega}_l = 0 \quad (3.1)$$

for $l = 1, 2, 3, \dots$ and either $\nu = \frac{3}{2}$, or a general ν given by (1.17). Since for the closed universe the variable $\mu = \cos \chi$ is real and $\mu \in (-1, +1)$, we are interested in Legendre functions $P_\nu^{l+\frac{1}{2}}(\mu)$, $Q_\nu^{l+\frac{1}{2}}(\mu)$ on the cut. (We emphasize here that the order μ of the Legendre function, e.g. in Q_ν^μ , has nothing to do with the independent variable μ .) Many useful formulas are given in [10]. In particular, we start from the general expression for Q_ν^μ in terms of a particular combination of two hypergeometric functions (given as the 4th expression on p. 168 in [10]), change $\theta \rightarrow \chi$ and put $\mu = l + \frac{1}{2}$. Employing then the well-known properties of hypergeometric functions for special values of parameters, we find, after some algebra, that functions $Q_\nu^{l+\frac{1}{2}}(\cos \chi)$ can be written as finite sums of the form

$$Q_{\nu}^{l+\frac{1}{2}}(\cos \chi) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{1/2} \chi} \frac{\Gamma(\frac{3}{2} + \nu + l)}{\Gamma(\frac{3}{2} + \nu)} \sum_{k=0}^l \frac{(-l)_k (l+1)_k}{(\frac{3}{2} + \nu)_k k!} \left(\frac{-1}{2 \sin \chi}\right)^k \cos \left[\left(k + \frac{1}{2} + \nu\right) \chi + (k+l+1) \frac{\pi}{2} \right], \quad (3.2)$$

where the standard notation $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ is used. In [10], p. 168, suitable expression in terms of hypergeometric functions is given also for Legendre functions P_{ν}^{μ} on the cut which one may wish to use as independent solutions. However, after putting $\mu = l + \frac{1}{2}$ and making arrangements similar to those above for $Q_{\nu}^{l+\frac{1}{2}}$, it turns out that $P_{\nu}^{l+\frac{1}{2}}$ diverge at $\chi = 0, \pi$. [For $l = 0, 1$ this was shown in Paper I.] Therefore, only functions $Q_{\nu}^{l+\frac{1}{2}}$ given by (3.2) will be appropriate objects in the following.

It is easy to check that (3.2) for $l = 0$ implies equation (I.4.9). A small calculation shows that (3.2) gives

$$Q_{n-\frac{1}{2}}^{3/2} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{3/2} \chi} \times \left\{ -(n+1) \sin \chi \cos(n\chi) + \sin[(n+1)\chi] \right\}, \quad (3.3)$$

which, after simple rearrangements, coincides with the first expression in equation (I.4.12) for $l = 1$. As in

Paper I, we have put $\nu = n - \frac{1}{2}$ in (3.3) and will do so hereafter.

With $(\delta T_{\varphi}^0)_l$ given and $k = +1$, the homogeneous equation corresponding to (1.15) becomes (3.1) with $\nu = \frac{3}{2}$, $n = 2$. We thus will consider solutions $Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}$, $n = 2$, determined by (3.2). In Paper I these solutions have been used for $l = 1$ and general n to determine $\omega_{l=1}$ when $\Omega_{l=1}$ is considered as a source. Since we also wish to consider solutions ω_l for Ω_l as a source with any l , we do not put $n = 2$ yet, rather give first explicitly few functions

$$\mathcal{Q}_n^l \stackrel{\text{def}}{=} Q_{n-\frac{1}{2}}^{l+\frac{1}{2}} \quad (3.4)$$

for general n and $l = 1, 2, \dots$, using the expression (3.2). As with $l = 1$, we shall also consider n purely imaginary, $n = iN$. Calculations become lengthy with l increasing. We have used MATHEMATICA for checking, simplifying and deriving some of the following formulas.

Starting first from (3.2) for $l = 1$, we get

$$\mathcal{Q}_n^1 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{3/2} \chi} [\cos \chi \sin(n\chi) - n \sin \chi \cos(n\chi)], \quad (3.5)$$

$$\mathcal{Q}_{iN}^1 = i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{3/2} \chi} [\cos \chi \sinh(N\chi) - N \sin \chi \cosh(N\chi)]. \quad (3.6)$$

For $l = 2$ we obtain

$$\mathcal{Q}_n^2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{5/2} \chi} \left\{ -\frac{3}{2} n \sin(2\chi) \cos(n\chi) + [2 - n^2 \sin^2 \chi + \cos(2\chi)] \sin(n\chi) \right\}, \quad (3.7)$$

$$\mathcal{Q}_{iN}^2 = i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{5/2} \chi} \left\{ -\frac{3}{2} N \sin(2\chi) \cosh(N\chi) + [2 + N^2 \sin^2 \chi + \cos(2\chi)] \sinh(N\chi) \right\}. \quad (3.8)$$

With $l = 3$ in (3.2) lengthier calculations lead to

$$\mathcal{Q}_n^3 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{7/2} \chi} \frac{1}{2} \left\{ n [(-n^2 - 11) \cos(2\chi) + n^2 - 19] \cos(n\chi) \sin \chi + 6 [(n^2 + 1) \cos(2\chi) - n^2 + 4] \cos \chi \sin(n\chi) \right\}, \quad (3.9)$$

$$\mathcal{Q}_{iN}^3 = i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{7/2} \chi} \frac{1}{2} \left\{ N [(N^2 - 11) \cos(2\chi) - N^2 - 19] \cosh(N\chi) \sin \chi + 6 [(-N^2 + 1) \cos(2\chi) + N^2 + 4] \cos \chi \sinh(N\chi) \right\}. \quad (3.10)$$

Finally, explicit expressions implied by (3.2) for $l = 4$ are

$$\begin{aligned} \mathcal{Q}_n^4 = & \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{9/2} \chi} \frac{1}{8} \left\{ 20n [(n^2 + 5) \cos(2\chi) - n^2 + 16] \cos(n\chi) \sin(2\chi) \right. \\ & \left. + [-3(n^4 - 25n^2 + 144) + 4(n^4 - 10n^2 - 96) \cos(2\chi) - (n^4 + 35n^2 + 24) \cos(4\chi)] \sin(n\chi) \right\}, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \mathcal{Q}_{iN}^4 = & i \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{9/2} \chi} \frac{1}{8} \left\{ 20N [(-N^2 + 5) \cos(2\chi) + N^2 + 16] \cosh(N\chi) \sin(2\chi) \right. \\ & \left. + [-3(N^4 + 25N^2 + 144) + 4(N^4 + 10N^2 - 96) \cos(2\chi) - (N^4 - 35N^2 + 24) \cos(4\chi)] \sinh(N\chi) \right\}. \end{aligned} \quad (3.12)$$

The behavior of these solutions at the poles $\chi = 0, \pi$, as well as the second independent solutions to the homogeneous Legendre equation (3.1), will be discussed below.

Now we have to distinguish the case when $(\delta T_\varphi^0)_l$ or Ω_l is considered as a source.

A. The angular momentum as a source for ω

In order to construct solutions to (1.15) for $k = +1$, we need solutions of the homogeneous Legendre equation (3.1) for $l = 1, 2, 3, \dots$ and $\nu = \frac{3}{2}$, $n = 2$, i.e. $\mathcal{Q}_2^l = \mathcal{Q}_{\frac{3}{2}}^{l+\frac{1}{2}}$. For $l = 1$ we have the solution (3.3) with $n = 2$:

$$\mathcal{Q}_2^1 = \mathcal{Q}_{\frac{3}{2}}^{\frac{3}{2}} = 2 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sin^{\frac{3}{2}} \chi. \quad (3.13)$$

This is well-behaved at both $\chi = 0$ and $\chi = \pi$, whereas the second (independent) solution

$$P_{\frac{3}{2}}^{\frac{3}{2}} = - \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\cos \chi}{\sin^{\frac{3}{2}} \chi} (1 + 2 \sin^2 \chi) \quad (3.14)$$

diverges at these poles. The solution \mathcal{Q}_2^1 implies [cf. equation (1.14)] $\omega_{l=1} = \omega_0 = \text{constant}$, independent of χ , as the solution of the original equation associated with (1.13). Such ω_0 may be time-dependent and may always enter ω in the case of closed universes, in which an “absolute rotation” does not occur. It cannot be removed (see equation (I.3.8), and text below). The other homogeneous solution for ω_1 , corresponding to $P_{3/2}^{3/2}$, diverges at the poles. For $l = 1$ the inhomogeneous equation (1.15) can then be integrated directly. The integration yields

$$\frac{d\omega_1}{d\chi} = \frac{2\kappa}{\sin^4 \chi} \int_0^\chi \sin^2 \chi' (\delta T_\varphi^0)_1 d\chi'. \quad (3.15)$$

Starting from the definition of the angular momentum $J(\chi_1, \chi_2)$ in (I.2.25), with $r = \sin \chi$, and integrating over θ we get

$$J(0, \chi) = \frac{8\pi}{3} \sqrt{\frac{3}{4\pi}} a^3 \int_0^\chi \sin^2 \chi' (\delta T_\varphi^0)_1 d\chi'. \quad (3.16)$$

Since (as before) $\omega_{l=1} = -\sqrt{\frac{4\pi}{3}} \omega$ and $\kappa = 8\pi$, equation (3.15) becomes the equation (I.3.6), and we thus recover the solution (I.3.8).

Turning now to $l \geq 2$ we are confronted with an intriguing problem. Direct calculations (checked and for $l = 4$ done by MATHEMATICA) show that for $n = 2$ all functions \mathcal{Q}_2^2 , \mathcal{Q}_2^3 , \mathcal{Q}_2^4 given by (3.7), (3.9) and (3.11) vanish. In other words, Legendre functions $Q_{3/2}^{5/2}$, $Q_{3/2}^{7/2}$, $Q_{3/2}^{9/2}$ do vanish on the cut! Using the recurrence relation (see [10], p.171, with $P_\nu^\mu \rightarrow Q_\nu^\mu$, and here we write x instead of $\mu = \cos \chi$)

$$\begin{aligned} Q_\nu^{\mu+2}(x) + 2(\mu+1) \frac{x}{(1-x^2)^{\frac{1}{2}}} Q_\nu^{\mu+1}(x) \\ + (\nu-\mu)(\nu+\mu+1) Q_\nu^\mu(x) = 0, \end{aligned} \quad (3.17)$$

we can prove that in fact all

$$Q_{3/2}^{l+\frac{1}{2}}(x) = 0 \quad \text{for } l \geq 2. \quad (3.18)$$

We are not aware of a mathematical reference where this is noticed.

How then to find regular solutions of the Legendre equation (3.1) for $\nu = \frac{3}{2}$, $n = 2$ and $l \geq 2$? We start from the following observations: if Q_ν^μ solves Legendre equation (3.1) but for some special values $\nu = \nu_0$ this Q_ν^μ vanishes, then the function $(\partial Q_\nu^\mu / \partial \nu)_{\nu=\nu_0}$ solves the equation for $\nu = \nu_0$ and may be non-vanishing. The proof can be done easily by a direct calculation. Therefore, we wish to calculate $\partial \mathcal{Q}_n^l / \partial n$ from the expressions (3.7), (3.9), (3.11), and put $n = 2$. Denoting

$$\tilde{\mathcal{Q}}_2^l = \frac{\partial \mathcal{Q}_n^l}{\partial n} \Big|_{n=2} = \frac{\partial Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}}{\partial n} \Big|_{n=2}, \quad (3.19)$$

we arrive at the following expressions:

$$\tilde{\mathcal{Q}}_2^2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{5/2} \chi} \left[-\frac{1}{4} \sin(4\chi) + 2 \sin(2\chi) - 3\chi \right], \quad (3.20)$$

$$\tilde{Q}_2^3 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{7/2}\chi} \times \left[\frac{1}{8}\sin(5\chi) - \frac{15}{8}\sin(3\chi) - 10\sin\chi + 15\chi\cos\chi\right], \quad (3.21)$$

$$\tilde{Q}_2^4 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{9/2}\chi} \left[-\frac{1}{8}\sin(6\chi) + 3\sin(4\chi) + \frac{375}{8}\sin(2\chi) - 15\chi[3\cos(2\chi) + 4]\right]. \quad (3.22)$$

MATHEMATICA yields nice expansions at $\chi \rightarrow 0$:

$$\tilde{Q}_2^2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-\frac{8}{5}\chi^{5/2} + \frac{2}{21}\chi^{9/2} + \mathcal{O}(\chi^{11/2})\right], \quad (3.23)$$

$$\tilde{Q}_2^3 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-\frac{8}{7}\chi^{7/2} - \frac{2}{21}\chi^{11/2} + \mathcal{O}(\chi^{15/2})\right], \quad (3.24)$$

$$\tilde{Q}_2^4 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-\frac{32}{21}\chi^{9/2} - \frac{24}{77}\chi^{13/2} + \mathcal{O}(\chi^{17/2})\right]. \quad (3.25)$$

Hence, functions (3.20)–(3.22) are well behaved near the origin,

$$\tilde{Q}_2^l \cong (\text{const.})\chi^{l+\frac{1}{2}} \quad \text{as } \chi \rightarrow 0, \quad (3.26)$$

however, they at first sight diverge at $\chi = \pi$.

Realizing that $\chi = \pi$ is an alternative origin, we take as independent solutions (notice that Legendre equation is invariant under $\mu \rightarrow -\mu$)

$$\tilde{\mathcal{P}}_2^l(\chi) = \tilde{Q}_2^l(\pi - \chi). \quad (3.27)$$

They read

$$\tilde{\mathcal{P}}_2^2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{5/2}\chi} \left[\frac{1}{4}\sin(4\chi) - 2\sin(2\chi) - 3(\pi - \chi)\right], \quad (3.28)$$

$$\begin{aligned} \tilde{\mathcal{P}}_2^3 &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{7/2}\chi} \left[\frac{1}{8}\sin(5\chi) - \frac{15}{8}\sin(3\chi) - \right. \\ &\quad \left. - 10\sin\chi - 15(\pi - \chi)\cos\chi\right], \end{aligned} \quad (3.29)$$

$$\begin{aligned} \tilde{\mathcal{P}}_2^4 &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\sin^{9/2}\chi} \left[\frac{1}{8}\sin(6\chi) - 3\sin(4\chi) - \right. \\ &\quad \left. - \frac{375}{8}\sin(2\chi) - 15(\pi - \chi)[3\cos(2\chi) + 4]\right]. \end{aligned} \quad (3.30)$$

At $\chi \rightarrow \pi$ they behave as (3.23)–(3.25) with $\chi \rightarrow \pi$, so

$$\tilde{\mathcal{P}}_2^l \cong (\text{const.})(\pi - \chi)^{l+\frac{1}{2}} \quad \text{as } \chi \rightarrow \pi. \quad (3.31)$$

The expansions (3.26) and (3.31) were obtained by direct calculations for $l = 2, 3, 4$. Let us briefly indicate a more general argument demonstrating their form for general l . In [10] one finds (see p.178) the recursion relations (here again $x = \cos\chi$)

$$xQ_\nu^\mu - Q_{\nu+1}^\mu = (\nu + \mu)(1 - x^2)^{\frac{1}{2}}Q_\nu^{\mu-1} \quad (3.32)$$

and

$$(1 - x^2)\frac{dQ_\nu^\mu}{dx} = (\nu + 1)xQ_\nu^\mu - (\nu - \mu + 1)Q_{\nu+1}^\mu. \quad (3.33)$$

Expressing $Q_{\nu+1}^\mu$ from the last equation, substituting into (3.32), taking $\partial/\partial\nu$ and putting $\nu = \frac{3}{2}, \mu = l + \frac{1}{2}$ and $x = \cos\chi$, one finds the recursion relation

$$\begin{aligned} \left(l + \frac{1}{2}\right)\cos\chi\frac{\partial Q_\nu^{l+\frac{1}{2}}}{\partial\nu}\bigg|_{3/2} + \sin\chi\frac{d}{d\chi}\frac{\partial Q_\nu^{l+\frac{1}{2}}}{\partial\nu}\bigg|_{3/2} \\ = (l+2)(l-2)\sin\chi\frac{\partial Q_\nu^{l-\frac{1}{2}}}{\partial\nu}\bigg|_{3/2}. \end{aligned} \quad (3.34)$$

Let us assume that for small χ one can write

$$\frac{\partial Q_\nu^{l+\frac{1}{2}}}{\partial\nu}\bigg|_{3/2} = \alpha_l\chi^p, \quad (3.35)$$

and that

$$\frac{\partial Q_\nu^{l-\frac{1}{2}}}{\partial\nu}\bigg|_{3/2} = \alpha_{l-1}\chi^{l-\frac{1}{2}} \quad (3.36)$$

is known. Then equation (3.34) implies $p = l + \frac{1}{2}$ and

$$\alpha_l = \frac{(l+2)(l-2)}{l+\frac{1}{2}}\alpha_{l-1}. \quad (3.37)$$

Continuing to $l = 2$ and regarding (3.20) for \tilde{Q}_2^2 , we obtain

$$\alpha_l = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(l+2)!(l-2)!}{(2l+1)!!}, \quad l \geq 2. \quad (3.38)$$

Therefore, the asymptotic expansion (3.26) can be written for general $l \geq 2$

$$\tilde{Q}_2^l \cong \alpha_l\chi^{l+\frac{1}{2}}, \quad (3.39)$$

where α_l is given by (3.38). It is easy to check that for $l = 2, 3, 4$ one gets the first terms in the expressions (3.23)–(3.25).

Next we need to calculate the Wronskians. Defining the Wronskians with respect to χ (rather than with respect to $\mu = \cos\chi$),

$$\mathcal{W}_l\{\tilde{Q}_2^l, \tilde{\mathcal{P}}_2^l\} = \tilde{Q}_2^l\frac{d\tilde{\mathcal{P}}_2^l}{d\chi} - \tilde{\mathcal{P}}_2^l\frac{d\tilde{Q}_2^l}{d\chi}, \quad (3.40)$$

and substituting from (3.20)–(3.22) and (3.28)–(3.30), MATHEMATICA yields:

$$\mathcal{W}_2 = -\frac{12\pi^2}{\sin\chi}, \quad \mathcal{W}_3 = -\frac{60\pi^2}{\sin\chi}, \quad \mathcal{W}_4 = -\frac{720\pi^2}{\sin\chi}. \quad (3.41)$$

Of course, the simple dependence on χ could have been obtained easily since the Wronskians satisfy a simple differential equation. MATHEMATICA is useful to determine the coefficients. (If the Wronskians are defined in

terms of $d/d \cos \chi$, the last expressions are multiplied by factor $-1/\sin \chi$.) Playing with recurrent relations given in [10] (p.171) for both $Q_\nu^\mu(x)$ and $Q_\nu^\mu(-x)$, and taking appropriate derivatives $\partial/\partial \nu$, we can find the general formula for \mathcal{W}_l , $l \geq 2$, in terms of $\tilde{\mathcal{Q}}_2^l$, $\tilde{\mathcal{P}}_2^l$ and $Q_{3/2}^{l-\frac{1}{2}}(-x)$:

$$\mathcal{W}_l = - (l+2)(l-2) \left(\tilde{\mathcal{Q}}_2^{l-1} \tilde{\mathcal{P}}_2^l + \tilde{\mathcal{Q}}_2^l \tilde{\mathcal{P}}_2^{l-1} \right) + 4 \left[\tilde{\mathcal{P}}_2^l Q_{3/2}^{l-\frac{1}{2}}(x) + \tilde{\mathcal{Q}}_2^l Q_{3/2}^{l-\frac{1}{2}}(-x) \right]. \quad (3.42)$$

Since $Q_{3/2}^{l+\frac{1}{2}} = 0$ for $l \geq 2$, the second term in square brackets does not vanish only for $l = 2$.

As before, we can now solve the inhomogeneous equation (1.15) with $k = +1$, $l = 2, 3, \dots$ by variation of the parameters. Remembering that $\tilde{\mathcal{Q}}_2^l$ is well-behaved at $\chi = 0$, whereas $\tilde{\mathcal{P}}_2^l$ at $\chi = \pi$, we write the solution in the form

$$\bar{\omega}_l = 2\kappa \left\{ \tilde{\mathcal{P}}_2^l \int_0^\chi \frac{\tilde{\mathcal{Q}}_2^l}{\mathcal{W}_l \sin^{\frac{1}{2}} \chi'} (\delta T_\varphi^0)_l d\chi' + \tilde{\mathcal{Q}}_2^l \int_\chi^\pi \frac{\tilde{\mathcal{P}}_2^l}{\mathcal{W}_l \sin^{\frac{1}{2}} \chi'} (\delta T_\varphi^0)_l d\chi' \right\}, \quad (3.43)$$

where for $l = 2, 3, 4$, functions $\tilde{\mathcal{Q}}_2^l$'s and $\tilde{\mathcal{P}}_2^l$'s are given by (3.20)–(3.22) and (3.28)–(3.30), and $(\mathcal{W}_l \sin^{\frac{1}{2}} \chi')^{-1} = (\text{const.}) \sin^{\frac{1}{2}} \chi'$ as a consequence of (3.41). For a general l , $\tilde{\mathcal{Q}}_2^l = (\partial Q_\nu^{l+\frac{1}{2}}/\partial \nu)_{\nu=3/2}$, $Q_\nu^{l+\frac{1}{2}}$ being determined by (3.2), and $\tilde{\mathcal{P}}_2^l(\chi) = \tilde{\mathcal{Q}}_2^l(\pi - \chi)$. The final physical solution is then obtained from $\omega_l = (\sin \chi)^{-3/2} \bar{\omega}_l$, with $\bar{\omega}_l$ given by (3.43). It is easy to make sure that for $\Omega - \omega$ regular at the poles (i.e., due to $(\delta T_\varphi^0)_l \sim \sin^2 \chi$ at the poles), ω_l is well-behaved at the poles.

Considering $(\delta T_\varphi^0)_l$ concentrated around the equator at $\chi \in (\pi/2 \pm \Delta)$, for example, then near the pole $\chi = 0$ we find ($l \geq 2$)

$$\omega_l = 2\kappa \alpha_l \chi^{l-1} \int_{\frac{\pi}{2}-\Delta}^{\frac{\pi}{2}+\Delta} \frac{\tilde{\mathcal{P}}_2^l}{\mathcal{W}_l \sin^{\frac{1}{2}} \chi'} (\delta T_\varphi^0)_l d\chi', \quad (3.44)$$

where factor α_l is given by (3.38).

B. The fluid angular velocity as a source

To solve (1.16) with $k = +1$ and Ω_l given, we use solutions (3.2) for general $\nu = n - \frac{1}{2}$. In particular, for $l = 1, \dots, 4$ we have explicit expressions (3.5)–(3.12) for both cases of n real and purely imaginary, $n = iN$. These solutions are well-behaved at the origin. Indeed at small

χ we find

$$\mathcal{Q}_n^1 = - \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{n}{3} (n^2 - 1) \chi^{3/2} + \mathcal{O}(\chi^{7/2}), \quad (3.45)$$

$$\mathcal{Q}_{iN}^1 = i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{N}{3} (N^2 + 1) \chi^{3/2} + \mathcal{O}(\chi^{7/2}), \quad (3.46)$$

$$\mathcal{Q}_n^2 = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{n}{15} (n^4 - 5n^2 + 4) \chi^{5/2} + \mathcal{O}(\chi^{9/2}), \quad (3.47)$$

$$\mathcal{Q}_{iN}^2 = i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{N}{15} (N^4 + 5N^2 + 4) \chi^{5/2} + \mathcal{O}(\chi^{9/2}), \quad (3.48)$$

$$\mathcal{Q}_n^3 = - \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{n}{105} (-n^6 + 14n^4 - 49n^2 + 36) \chi^{7/2} + \mathcal{O}(\chi^{11/2}), \quad (3.49)$$

$$\mathcal{Q}_{iN}^3 = -i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{N}{105} (N^6 + 14N^4 + 49N^2 + 36) \chi^{7/2} + \mathcal{O}(\chi^{11/2}), \quad (3.50)$$

$$\mathcal{Q}_n^4 = - \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{n}{945} (n^8 - 30n^6 + 273n^4 - 820n^2 + 576) \chi^{9/2} + \mathcal{O}(\chi^{13/2}), \quad (3.51)$$

$$\mathcal{Q}_{iN}^4 = -i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{N}{945} (N^8 + 30N^6 + 273N^4 + 820N^2 + 576) \chi^{9/2} + \mathcal{O}(\chi^{13/2}). \quad (3.52)$$

To obtain these expansions we could not have used formulas given, for example, in [10] (see p.196), since the leading term in $Q_\nu^\mu(x)$ at $x = 1$ is proportional to $\cos(\pi\mu)$; hence, it vanishes for $\mu = l + \frac{1}{2}$. However, we may proceed similarly to the way how we obtained the coefficients α_l in (3.38). We substitute for $Q_{\nu+1}^\mu(x)$ in (3.32) from equation (3.33), put $x = \cos \chi$, $\mu = l + \frac{1}{2}$, $\nu = n - \frac{1}{2}$, $Q_{n-\frac{1}{2}}^{l+\frac{1}{2}} = \mathcal{Q}_n^l$, and so arrive at the recursion relation

$$\mathcal{Q}_n^l \frac{l + \frac{1}{2}}{n - l} + \tan \chi \frac{1}{n - l} \frac{d\mathcal{Q}_n^l}{d\chi} + (l + n) \tan \chi \mathcal{Q}_n^{l-1} = 0. \quad (3.53)$$

Regarding results (3.45)–(3.52), assume that for small χ

$$\mathcal{Q}_n^l = \alpha_{l,n} \chi^p, \quad (3.54)$$

and that $\mathcal{Q}_n^{l-1} = \alpha_{l-1,n} \chi^{l-\frac{1}{2}}$ is known. Then (3.53) implies $p = l + \frac{1}{2}$ and

$$\alpha_{l,n} = - \frac{(n+l)(n-l)}{2l+1} \alpha_{l-1,n}. \quad (3.55)$$

Continuing to $l = 1$ we get

$$\alpha_{l,n} = (-1)^l \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{n(n^2 - l^2)[n^2 - (l-1)^2] \cdots (n^2 - 4)(n^2 - 1)}{(2l+1)!!}. \quad (3.56)$$

For $n = iN$,

$$\mathcal{Q}_{iN}^l = \alpha_{l,iN} \chi^{l+\frac{1}{2}}, \quad (3.57)$$

where

$$\alpha_{l,iN} = i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \times \frac{N(N^2 + l^2)[N^2 + (l-1)^2] \cdots (N^2 + 4)(N^2 + 1)}{(2l+1)!!}. \quad (3.58)$$

The second (independent) solutions which are well-behaved at the other pole, $\chi = \pi$, are, as before, obtained by taking \mathcal{Q} 's at $\pi - \chi$. Thus,

$$\mathcal{P}_n^l(\chi) = \mathcal{Q}_n^l(\pi - \chi) \quad , \quad \mathcal{P}_{iN}^l(\chi) = \mathcal{Q}_{iN}^l(\pi - \chi) \quad (3.59)$$

are solutions to the homogeneous Legendre equation (3.1) on the cut which are regular at $\chi = \pi$.

We do not write them down explicitly since they follow easily from \mathcal{Q} 's given in (3.5)–(3.12). At $\chi \rightarrow \pi$

$$\mathcal{P}_n^l = \alpha_{l,n}(\pi - \chi)^{l+\frac{1}{2}} \quad , \quad \mathcal{P}_{iN}^l = \alpha_{l,iN}(\pi - \chi)^{l+\frac{1}{2}}. \quad (3.60)$$

What are the Wronskians? We get

$$\mathcal{W}_{1,n}\{\mathcal{Q}_n^1, \mathcal{P}_n^1\} = \frac{\pi}{2}n(n^2 - 1)\frac{\sin(n\pi)}{\sin \chi}, \quad (3.61)$$

$$\mathcal{W}_{1,N}\{\mathcal{Q}_{iN}^1, \mathcal{P}_{iN}^1\} = \frac{\pi}{2}N(N^2 + 1)\frac{\sinh(N\pi)}{\sin \chi}, \quad (3.62)$$

$$\mathcal{W}_{2,n}\{\mathcal{Q}_n^2, \mathcal{P}_n^2\} = -\frac{\pi}{2}n(n^4 - 5n^2 + 4)\frac{\sin(n\pi)}{\sin \chi}, \quad (3.63)$$

$$\mathcal{W}_{2,N}\{\mathcal{Q}_{iN}^2, \mathcal{P}_{iN}^2\} = \frac{\pi}{2}N(N^4 + 5N^2 + 4) \times \frac{\sinh(N\pi)}{\sin \chi}, \quad (3.64)$$

where we restricted ourselves to $l = 1, 2$ since the polynomials depending on n (resp. N) are proportional to those in the expansions (3.45)–(3.52) at $\chi \rightarrow 0$ for which we succeeded to derive the formulas (3.57)–(3.59) for general l . (This is due to the fact that the functional dependence of the Wronskian is governed by a simple differential equation, and the proportionality factor can be determined at any point, e.g., at $\chi \rightarrow 0$.) The general form of the Wronskians can be derived by expressing functions \mathcal{P}_n^l in terms of Q_ν^μ 's and P_ν^μ 's. Using the relation giving $Q_\nu^\mu(-x)$ in terms of $Q_\nu^\mu(x)$ and $P_\nu^\mu(x)$ [not to be confused with $\mathcal{P}_n^l = \mathcal{Q}_n^l(\pi - \chi)$] – see e.g. [10], p.170 – and putting $\mu = l + \frac{1}{2}$, $\nu = n - \frac{1}{2}$, we get

$$\begin{aligned} \mathcal{P}_n^l &= Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}(-x) = (-1)^{l+1} \times \\ &\times \left[Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}(x) \cos(n\pi) + \frac{\pi}{2} P_{n-\frac{1}{2}}^{l+\frac{1}{2}}(x) \sin(n\pi) \right]. \end{aligned} \quad (3.65)$$

Employing then the well known formula ([10], p.170)

$$\mathcal{W}[P_\nu^\mu, Q_\nu^\mu] = \frac{\Gamma(1 + \nu + \mu)}{\Gamma(1 + \nu - \mu)} \frac{1}{1 - x^2}, \quad (3.66)$$

we arrive at

$$\begin{aligned} \mathcal{W}_{l,n}[\mathcal{Q}_n^l, \mathcal{P}_n^l] &= (-1)^{l+1} \frac{\pi}{2} n(n^2 - l^2) \times \\ &\times [n^2 - (l-1)^2] \cdots (n^2 - 4)(n^2 - 1) \frac{\sin(n\pi)}{\sin \chi}, \end{aligned} \quad (3.67)$$

and

$$\begin{aligned} \mathcal{W}_{l,N}[\mathcal{Q}_{iN}^l, \mathcal{P}_{iN}^l] &= \frac{\pi}{2} N(N^2 + l^2) \times \\ &\times [N^2 + (l-1)^2] \cdots (N^2 + 4)(N^2 + 1) \times \frac{\sinh(N\pi)}{\sin \chi}. \end{aligned} \quad (3.68)$$

It is easy to regain (3.61)–(3.64) for $l = 1, 2$.

Now we can finally solve the inhomogeneous equation (1.16), with Ω_l as a source, again by the variation of the parameters. Regarding the relation (1.14) between $\bar{\omega}_l$ and ω_l we arrive at the solutions in the form

$$\begin{aligned} \omega_l &= \frac{n^2 - 4}{(\sin \chi)^{3/2}} \left\{ \mathcal{P}_n^l \int_0^\chi \frac{(\sin \chi')^{3/2}}{\mathcal{W}_{l,n}} \mathcal{Q}_n^l \Omega_l d\chi' + \right. \\ &\quad \left. \mathcal{Q}_n^l \int_\chi^\pi \frac{(\sin \chi')^{3/2}}{\mathcal{W}_{l,n}} \mathcal{P}_n^l \Omega_l d\chi' \right\}, \end{aligned} \quad (3.69)$$

analogously for $n = iN$:

$$\begin{aligned} \omega_l &= -\frac{N^2 + 4}{(\sin \chi)^{3/2}} \left\{ \mathcal{P}_{iN}^l \int_0^\chi \frac{(\sin \chi')^{3/2}}{\mathcal{W}_{l,N}} \mathcal{Q}_{iN}^l \Omega_l d\chi' + \right. \\ &\quad \left. + \mathcal{Q}_{iN}^l \int_\chi^\pi \frac{(\sin \chi')^{3/2}}{\mathcal{W}_{l,N}} \mathcal{P}_{iN}^l \Omega_l d\chi' \right\}. \end{aligned} \quad (3.70)$$

All quantities entering (3.69) and (3.70) are given above (for $l = 1, \dots, 4$ in the explicit forms), the time-dependent ‘constants’ n, N are determined by (1.17), which was used to express λ^2 in terms of n^2 and N^2 . Notice that solutions (3.70) are all real since \mathcal{Q}_{iN}^l and \mathcal{P}_{iN}^l are purely imaginary. We could have divided all solutions of the homogeneous equation given in the present section by n (resp. N) as we did for $l = 1$ in Paper I, and so define $^*\mathcal{Q}_n^l = \mathcal{Q}_n^l/n$, $^*\mathcal{Q}_{iN}^l = \mathcal{Q}_{iN}^l/N$ etc which, as can easily be seen, remain non-vanishing for $n = 0, N = 0$. However, the Wronskians (3.67) and (3.68) are proportional to n^2 (resp. N^2) as $n \rightarrow 0$ (resp. $N \rightarrow 0$), so the expressions $\mathcal{Q}_n^l \mathcal{P}_n^l / \mathcal{W}_{l,n}$ (the same with N) remain non-vanishing in this limit. Hence physical solutions (3.69), (3.70) for ω_l do not vanish when $n = 0$ (resp. $N = 0$).

Now the final solutions (3.69), (3.70) for any l can be applied to the special situations considered for $l = 1$ in Paper I. So, for large N (3.68) implies

$$\mathcal{W}_{l,N} = \frac{\pi}{2} N^{2l+1} \frac{e^{N\pi}}{2 \sin \chi}, \quad (3.71)$$

and the inspection of (3.5)–(3.12) and the relation (3.60) lead to

$$\mathcal{P}_{iN}^l = i(-1)^l \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{1}{2} \frac{N^l e^{N(\pi - \chi)}}{\sin^{1/2} \chi}. \quad (3.72)$$

For N large but χ small expansions (3.45)–(3.52) and general formulas (3.57), (3.59) give

$$\mathcal{Q}_{iN}^l = i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{N^{2l+1}}{(2l+1)!!} \chi^{l+\frac{1}{2}}. \quad (3.73)$$

Hence our solution (3.70) for N large near the origin is

$$\omega_l = \frac{N^{l+2}\chi^{l-1}}{(2l+1)!!} \int_{\chi}^{\pi} \Omega_l(\chi') \sin^2 \chi' e^{-N\chi'} d\chi'. \quad (3.74)$$

For $l = 1$ we recover the leading term (independent of χ) in the expression (I.4.28). The next term in front of the integral in (3.74), which is $\sim \chi^{l+1}$, comes from the second terms in the expansions (3.45)–(3.52) indicated only by $\mathcal{O}(\chi^{l+\frac{5}{2}})$; their exact forms, if needed, can be obtained without difficulty by MATHEMATICA. The resulting ω_l at given small χ can become large at high l 's since $N^2 = \kappa a^2(\rho + p) - 4$ may become very large in, say, radiation dominated universes (when $(\rho + p) \sim a^{-4}$) close to big bang. Near the perturbation with Ω_l concentrated in $r_0(\chi_0) \pm \Delta$, with $N\Delta \ll 1$, N large, the solution (3.70) leads to

$$\omega_l(\chi_0) = \frac{1}{2} \int_0^{\pi} N e^{-N|\chi' - \chi_0|} \Omega_l(\chi') \left(\frac{\sin \chi'}{\sin \chi_0} \right)^2 d\chi', \quad (3.75)$$

i.e., the result identical for all l , and thus coinciding for $l = 1$ with (I.4.29), in which $\lambda^2 = N^2 + 4$, and only the leading term in N is kept.

IV. AXISYMMETRIC TOROIDAL PERTURBATIONS FOR $k = -1$

With $k = -1$, the homogeneous differential equation corresponding to the equation (1.16) with Ω_l as a source

takes the form

$$\frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial \bar{\omega}_l}{\partial \mu} \right] + \left[\nu(\nu + 1) - \frac{(l + \frac{1}{2})^2}{1 - \mu^2} \right] \bar{\omega}_l = 0, \quad (4.1)$$

again the Legendre equation. Now $\mu = \sqrt{1 + r^2} = \cosh \chi$, $r = \sinh \chi$, so the independent variable $\mu \in (1, +\infty)$. The solutions of the Legendre equation for μ such that its real part is ≥ 1 (μ is then often written as z) are sometimes denoted by capital Gothic script letters. This is the case in [10] from which we shall again take some formulas but we keep the Latin script. If $\nu = \frac{3}{2}$, equation (4.1) corresponds to the inhomogeneous equation (1.15) with $(\delta T_{\varphi}^0)_l$ as a source.

As in the case $k = +1$, we start from the general expression for the solution in terms of the combination of two hypergeometric functions. We first consider the Legendre function $P_{\nu}^{\mu}(z)$ as given in [10] (p. 154, the fourth line), substitute for $z = \mu = \cosh \chi$, put $\mu = l + \frac{1}{2}$ and, as before, $\nu + \frac{1}{2} = n$. Regarding standard properties of the hypergeometric functions for special values of parameters, we find after simple algebra that functions $P_{n-\frac{1}{2}}^{l+\frac{1}{2}}(\cosh \chi)$ can be rewritten as the following finite sums:

$$P_{n-\frac{1}{2}}^{l+\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(n+l+1)}{\Gamma(n+1)} \frac{1}{(\sinh \chi)^{\frac{1}{2}}} \left\{ \sum_{k=0}^l \frac{(-l)_k (l+1)_k}{(n+1)_k k!} \left[\frac{1}{(2 \sinh \chi)^k} \left(e^{(k+n)\chi} + (-1)^{l+k} e^{-(k+n)\chi} \right) \right] \right\}, \quad (4.2)$$

(cf. (3.2)). We shall also use

$$P_{n-\frac{1}{2}}^{-(l+\frac{1}{2})} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(n-l)}{\Gamma(n+1)} \frac{1}{(\sinh \chi)^{\frac{1}{2}}} \left\{ \sum_{k=0}^l \frac{(-l)_k (l+1)_k}{(n+1)_k k!} \left[\frac{1}{(2 \sinh \chi)^k} \left(e^{(k+n)\chi} - (-1)^{l+k} e^{-(k+n)\chi} \right) \right] \right\}, \quad (4.3)$$

which is also the solution since the Legendre equation is invariant under the change $l + \frac{1}{2} \rightarrow -(l + \frac{1}{2})$. This solution can be written as a linear combination of $P_{n-\frac{1}{2}}^{l+\frac{1}{2}}$ and of another solution, $Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}$, as follows[12]

$$P_{n-\frac{1}{2}}^{-(l+\frac{1}{2})} = \frac{\Gamma(n-l)}{\Gamma(n+l+1)} \left(P_{n-\frac{1}{2}}^{l+\frac{1}{2}} + \frac{2}{\pi} i Q_{n-\frac{1}{2}}^{l+\frac{1}{2}} \right). \quad (4.4)$$

The last equation combined with (4.2) and (4.3) can be

used to find a suitable form of the solutions $Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}$:

$$Q_{n-\frac{1}{2}}^{l+\frac{1}{2}} = i \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{(n+l)(n+l-1) \cdots (n+1)}{(\sinh \chi)^{\frac{1}{2}}} \times \sum_{k=0}^l \frac{(-1)^{l+k} (-l)_k (l+1)_k}{(n+1)_k k!} \frac{e^{-(k+n)\chi}}{(2 \sinh \chi)^k}. \quad (4.5)$$

Another useful expression for $P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}$ is provided by the inversion of Whipple's formula (see e.g. [10], p. 164)

which in our case of μ and ν reads

$$P_{n-\frac{1}{2}}^{-(l+\frac{1}{2})} \left(\frac{z}{\sqrt{z^2-1}} \right) = \frac{(z^2-1)^{\frac{1}{4}} e^{in\pi} Q_l^{-n}(z)}{\sqrt{\pi/2} \Gamma(l-n+1)}, \quad (4.6)$$

where $z = \cosh \chi$ (notice that the orders and degrees in P and Q are here interchanged).

The case $k = -1$ is simpler than that with $k = +1$. The “constant” parameter $n = \nu + 1/2$ does not become imaginary (cf. equation (1.17)), and the particular case $\nu = 3/2$, $n = 2$, corresponding to the homogeneous equation in (1.15) with δT_φ^0 as a source, does not require as special procedure as for $k = +1$. The relevant independent solutions of the homogeneous equations are

$$\overline{\mathcal{P}}_n^l = P_{n-\frac{1}{2}}^{-(l+\frac{1}{2})}, \quad \overline{\mathcal{Q}}_n^l = -i Q_{n-\frac{1}{2}}^{l+\frac{1}{2}}, \quad (4.7)$$

where we use the notation similar to that with $k = +1$ but put bars above \mathcal{P} 's and \mathcal{Q} 's to distinguish the two cases. The solutions (4.7) are given as finite sums by (4.3), (4.5), respectively (4.6).

The asymptotic behavior of $\overline{\mathcal{Q}}_n^l$ and $\overline{\mathcal{P}}_n^l$ at the origin and at infinity can be read off from literature where

asymptotic forms of P_ν^μ and Q_ν^μ are given (e.g. [10], p. 196 and 197). After arrangements we find at small χ

$$\overline{\mathcal{P}}_n^l = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{\chi^{l+\frac{1}{2}}}{(2l+1)!!}, \quad (4.8)$$

$$\overline{\mathcal{Q}}_n^l = (-1)^{l+1} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{(2l-1)!!}{\chi^{l+\frac{1}{2}}}, \quad (4.9)$$

at large χ

$$\overline{\mathcal{P}}_n^l = \frac{1}{\sqrt{\pi}} \frac{(2 \cosh \chi)^{n-\frac{1}{2}}}{(n+l)(n+l-1) \cdots n}, \quad (4.10)$$

$$\overline{\mathcal{Q}}_n^l = (-1)^l \sqrt{\pi} \frac{(n+l)(n+l-1) \cdots (n+1)}{(2 \cosh \chi)^{n+\frac{1}{2}}}. \quad (4.11)$$

Before we consider the solutions of the inhomogeneous equations we list explicit expressions for $\overline{\mathcal{P}}_n^l$ and $\overline{\mathcal{Q}}_n^l$ for $l = 1, 2, 3$ as they follow, after suitable arrangements, from general formulas (4.3), (4.5) and the definition (4.7). They look as follows:

$$\overline{\mathcal{P}}_n^1 = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{2n(n^2-1)(\sinh \chi)^{3/2}} \left\{ (n-1) \sinh[(n+1)\chi] - (n+1) \sinh[(n-1)\chi] \right\}, \quad (4.12)$$

$$\overline{\mathcal{Q}}_n^1 = - \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{1}{2(\sinh \chi)^{3/2}} \left[(n+1)e^{-(n-1)\chi} - (n-1)e^{-(n+1)\chi} \right], \quad (4.13)$$

$$\begin{aligned} \overline{\mathcal{P}}_n^2 = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{n(n^2-1)(n^2-4)(\sinh \chi)^{5/2}} & \left\{ (n+2) \sinh \chi [-3 \cosh\{(n+1)\chi\} \right. \\ & \left. + (n+1) \sinh \chi \sinh(n\chi)] + 3 \sinh[(n+2)\chi] \right\}, \end{aligned} \quad (4.14)$$

$$\overline{\mathcal{Q}}_n^2 = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{1}{2(\sinh \chi)^{5/2}} \left\{ (n+2) \sinh \chi \left[(n+1) \sinh \chi e^{-n\chi} + 3e^{-(n+1)\chi} \right] + 3e^{-(n+2)\chi} \right\}, \quad (4.15)$$

$$\begin{aligned} \overline{\mathcal{P}}_n^3 = \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \frac{1}{n(n^2-1)(n^2-4)(n^2-9)(\sinh \chi)^{7/2}} & \left\{ (n+3) \sinh \chi \left[15 \cosh\{(n+2)\chi\} \right. \right. \\ & \left. \left. + (n+2) \sinh \chi [(n+1) \cosh(n\chi) \sinh(n\chi) - 6 \sinh\{(n+1)\chi\}] - 15 \sinh\{(n+3)\chi\} \right] \right\}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \overline{\mathcal{Q}}_n^3 = - \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \frac{1}{(\sinh \chi)^{7/2}} & \left\{ (n+3) \sinh \chi \left[15e^{-(n+2)\chi} + (n+2) \sinh \chi \times \right. \right. \\ & \left. \left. \times [(n+1) \sinh \chi e^{-n\chi} + 6e^{-(n+1)\chi}] \right] + 15e^{-(n+3)\chi} \right\}. \end{aligned} \quad (4.17)$$

A. The angular momentum as a source of ω

We need to solve the homogeneous equation (4.1) for $\nu = \frac{3}{2}$, $n = 2$. At first sight we again encounter a peculiar case as with $k = +1$, since $\Gamma(n - l)$ in (4.3) has a pole at $l = 2$ for $n = 2$, the expression (4.4) appears to have also a pole in this case, and explicit formulas (4.14) and (4.16) seem to diverge for $n = 2$. However, Whipple's formula (4.6) for $\overline{\mathcal{P}}_n^l$ and the following formula

$$\overline{\mathcal{P}}_n^l(\tilde{z}) = \frac{1}{\sqrt{\pi/2}} \frac{e^{-in\pi}}{\Gamma(l+n+1)} \frac{1}{(\tilde{z}^2 - 1)^{1/4}} Q_l^n(\tilde{z}/\sqrt{\tilde{z}^2 - 1}), \quad (4.18)$$

which, after employing the relation between Q_l^{-n} and Q_l^n [10], is its consequence, show no peculiar behavior for $n = 2$. In fact, the last formula can be used to obtain simple explicit expressions for $\overline{\mathcal{P}}_n^l$ because (e.g. [8], formula 8.6.7)

$$Q_l^2(z) = (z^2 - 1) \frac{d^2 Q_l}{dz^2}, \quad (4.19)$$

where

$$Q_l(z) = \frac{1}{2} P_l(z) \log \frac{z+1}{z-1} + W_{l-1}(z), \quad (4.20)$$

W_{l-1} are polynomials of degree $l - 1$ given in terms of Legendre polynomials $P_l(z)$.

The same result can be derived directly from the original expressions (4.3) and (4.5) by taking the limit $n \rightarrow 2$. So, we take as our solutions of the homogeneous equations (4.1) for $\nu = 3/2$, $n = 2$ the functions

$$\overline{\mathcal{P}}_2^l = \lim_{n \rightarrow 2} \overline{\mathcal{P}}_n^l, \quad \overline{\mathcal{Q}}_2^l = \lim_{n \rightarrow 2} \overline{\mathcal{Q}}_n^l, \quad (4.21)$$

where $\overline{\mathcal{P}}_n^l$ and $\overline{\mathcal{Q}}_n^l$ are given by (4.7), (4.3) and (4.5). The behavior of $\overline{\mathcal{P}}_2^l$ and $\overline{\mathcal{Q}}_2^l$ at $\chi \rightarrow \infty$ is given by (4.10) and (4.11) with $n = 2$; the first term in the expansions of $\overline{\mathcal{P}}_n^l$ and $\overline{\mathcal{Q}}_n^l$ at $\chi \rightarrow 0$, equations (4.8) and (4.9), does not depend on n , so $\overline{\mathcal{P}}_2^l$ at $\chi \rightarrow 0$ behaves as $\overline{\mathcal{P}}_n^l$.

The explicit expressions for $l = 1, 2, 3$ follow easily from taking the limits $n \rightarrow 2$ in (4.12)–(4.17). One finds

$$\overline{\mathcal{P}}_2^1 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{3} (\sinh \chi)^{3/2}, \quad (4.22)$$

$$\overline{\mathcal{Q}}_2^1 = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{(\sinh \chi)^{3/2}} [\cosh \chi (1 - 2 \sinh^2 \chi) + 2 \sinh^3 \chi], \quad (4.23)$$

$$\overline{\mathcal{P}}_2^2 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(\sinh \chi)^{5/2}} \frac{1}{96} [12\chi - 8 \sinh(2\chi) + \sinh(4\chi)], \quad (4.24)$$

$$\overline{\mathcal{Q}}_2^2 = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{3}{(\sinh \chi)^{5/2}}, \quad (4.25)$$

$$\overline{\mathcal{P}}_2^3 = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(\sinh \chi)^{7/2}} \frac{1}{480} [120\chi \cosh \chi - 80 \sinh \chi - 15 \sinh(3\chi) + \sinh(5\chi)], \quad (4.26)$$

$$\overline{\mathcal{Q}}_2^3 = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{(\sinh \chi)^{7/2}} \left\{ 5 \sinh \chi [e^{-4\chi} + 4 \sinh \chi (3 \sinh \chi e^{-2\chi} + 6 e^{-3\chi})] + 15 e^{-5\chi} \right\}. \quad (4.27)$$

Starting from the well-known formula (e.g. [10], p.165) for the Wronskian $\mathcal{W}\{P_\nu^{-\mu}, Q_\nu^\mu\} = e^{i\pi\mu}/(1 - z^2)$, $z = \cosh \chi$, which is independent of ν and, in our case with $\mu = l + \frac{1}{2}$, the value of l changes only the sign of \mathcal{W} , we easily get

$$\overline{\mathcal{W}}_{l,n}\{\overline{\mathcal{P}}_n^l, \overline{\mathcal{Q}}_n^l\} = \overline{\mathcal{W}}_l\{\overline{\mathcal{P}}_2^l, \overline{\mathcal{Q}}_2^l\} = \frac{(-1)^{l+1}}{\sinh \chi}. \quad (4.28)$$

Employing again the method of variation of parameters and regarding the asymptotic behavior of $\overline{\mathcal{P}}_n^l$ and $\overline{\mathcal{Q}}_n^l$, we arrive at the well-behaved solutions of the inhomogeneous equation (1.16) – and hence of the original equation (1.14) – in the form

$$\omega_l = \frac{2\kappa}{(\sinh \chi)^{3/2}} \left[\overline{\mathcal{Q}}_2^l \int_0^\chi \frac{\overline{\mathcal{P}}_2^l(\delta T_\varphi^0)_l}{\overline{\mathcal{W}}_l(\sinh \chi')^{\frac{1}{2}}} d\chi' + \overline{\mathcal{P}}_2^l \int_\chi^\infty \frac{\overline{\mathcal{Q}}_2^l(\delta T_\varphi^0)_l}{\overline{\mathcal{W}}_l(\sinh \chi')^{\frac{1}{2}}} d\chi' \right]. \quad (4.29)$$

For $l = 1$ the angular momentum in $\langle 0, \chi \rangle$ is given by equation (3.16), with $\sin \chi' \rightarrow \sinh \chi'$; ω_1 and $(\delta T_\varphi^0)_1$ are connected with ω and δT_φ^0 as in the cases $k = 0, +1$. Noticing that $\overline{\mathcal{W}}$ defined in (I.3.13), is related to $\overline{\mathcal{Q}}_2^1$ given in (4.23) by $\overline{\mathcal{W}} = -\sqrt{2/\pi} (\sinh \chi)^{-3/2} \overline{\mathcal{Q}}_2^1$, we easily find that for $l = 1$ the solution (4.29) goes over into the solution (I.3.14).

Next consider $(\delta T_\varphi^0)_l$ concentrated at $\chi \in (\chi_0 \pm \Delta)$, $\chi_0 > 0$. Then near the origin $\chi \rightarrow 0$

$$\omega_l = 2\kappa \overline{\alpha}_l \chi^{l-1} \int_{\chi_0 - \Delta}^{\chi_0 + \Delta} \overline{\mathcal{Q}}_2^l (-1)^{l+1} (\sinh \chi')^{\frac{1}{2}} (\delta T_\varphi^0)_l d\chi', \quad (4.30)$$

where (see (4.8))

$$\overline{\alpha}_l = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(2l+1)!!}. \quad (4.31)$$

B. The fluid angular velocity as a source

To solve the inhomogeneous equation (1.16) with $k = -1$ and Ω_l given, we use solutions $\overline{\mathcal{P}}_n^l$ and $\overline{\mathcal{Q}}_n^l$ determined by (4.7), (4.3) and (4.5). For $l = 1, 2, 3$ they are given in explicit, arranged forms by (4.12)–(4.17). The solution $\overline{\omega}_l$, obtained by variation of the parameters, leads,

regarding (1.15), to the following physical solution well-behaved at the origin and vanishing at infinity:

$$\omega_l = \frac{1}{(\sinh \chi)^{3/2}} \left[\overline{\mathcal{Q}}_n^l \int_0^\chi \frac{\overline{\mathcal{P}}_n^l (-\lambda^2 \Omega_l) (\sinh \chi')^{3/2}}{\overline{\mathcal{W}}_{l,n}} d\chi' + \overline{\mathcal{P}}_n^l \int_\chi^\infty \frac{\overline{\mathcal{Q}}_n^l (-\lambda^2 \Omega_l) (\sinh \chi')^{3/2}}{\overline{\mathcal{W}}_{l,n}} d\chi' \right]. \quad (4.32)$$

Substituting for $\overline{\mathcal{W}}_{l,n}$ from (4.28) and for $\lambda^2(t) = n^2(t) - 4$ as it follows from (1.17) with $k = -1$, we finally have

$$\omega_l = \frac{(-1)^l (n^2 - 4)}{(\sinh \chi)^{3/2}} \left\{ \overline{\mathcal{Q}}_n^l \int_0^\chi \overline{\mathcal{P}}_n^l \Omega_l (\sinh \chi')^{5/2} d\chi' + \overline{\mathcal{P}}_n^l \int_\chi^\infty \overline{\mathcal{Q}}_n^l \Omega_l (\sinh \chi')^{5/2} d\chi' \right\}. \quad (4.33)$$

For $l = 1$ we notice that with $\overline{\mathcal{Q}}_n^1$ and $\overline{\mathcal{P}}_n^1$ we regain the solution (I.4.34).

Assume Ω_l non-vanishing at large χ . Then the solution (4.33) and the asymptotic results for $\overline{\mathcal{Q}}_n^1$ and $\overline{\mathcal{P}}_n^1$, equations (4.8)–(4.11), imply that at small χ

$$\omega_l = \frac{(n^2 - 4)(n + l)(n + l - 1) \cdots (n + 1)}{4(2l + 1)!!} \chi^{l-1} \times \int_\chi^\infty (\sinh \chi')^2 e^{-n\chi'} \Omega_l d\chi'. \quad (4.34)$$

Close to the perturbation,

$$\omega_l = \frac{1}{2} \frac{(n^2 - 4)}{n} \int_0^\infty \left(\frac{\sinh \chi'}{\sinh \chi_0} \right)^2 e^{-n|\chi' - \chi_0|} \Omega_l d\chi'. \quad (4.35)$$

Here l does enter the result directly. So for $l = 1$ it goes over immediately to (I.4.38). It is also easy to see that (4.34) gives for $l = 1$ the first terms in (I.4.37).

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